Chapter 1

Formulating (setting-up) Linear Programming Problems

When we use the term "linear programming (LP) problem" we mean the following:

1. a linear expression called the **objective function** which describes a quantity to be maximized or minimized;
2. one or more linear inequalities (or equalities) called **constraints** which restrict the allowed possible solutions; and
3. **nonnegativity conditions** on the variables. (These are occasionally omitted from some LP problems.)

Here is an example of an LP problem:

max	$P = x_1 + 2x_2$	objective function
if	$x_1 + 3x_2 \leq 4$	first constraint
	$2x_1 - x_2 \geq 3$	second constraint
	$x_1 \geq 0, \quad x_2 \geq 0$	nonnegativity conditions

Note that we denote variables by the letter x with different numerical subscripts. Maximum will be denoted "max" while minimum will be denoted "min".

In practice, we are not given such an LP problem to solve directly. Instead we must first translate verbal statements into an LP problem. This translation process involves not only the three parts mentioned above, but also a *variable dictionary* that relates the variables to the verbal statement and gives meaning to the variables.

EXAMPLE 1:

The Handy Company has determined that a typical secretary can type 8 pages and file 20 reports per hour while a typical clerk can type 4 pages and file 40 reports per hour. After extensive research, the company has determined that it needs 320 pages typed and 2600 reports filed each hour. If secretaries are paid \$11 per hour and clerks are paid \$5 per hour, how many secretaries and clerks should the Handy Company hire to meet its needs at a minimum cost?

Problem formulation: We want to formulate an LP problem describing this situation. Let us itemize the key steps in the process.

Step 1. Determine the variables and list them with nonnegativity conditions:
Usually buried in an LP problem statement is a key phrase which asks us to determine the values of certain items in order to maximize or minimize something. The items with the unknown values which we are to determine are our desired (decision) variables. They should be named x with an appropriate subscript. In this example, the key phrase is "how many secretaries and clerks should be hired?"

The number of secretaries is an unknown whose value we want to determine. We name it x_1. Similarly, the number of clerks is an unknown we want to determine. We name it x_2. We try to make these variable definitions as precise as possible. For example, we say "x_1 = the number of secretaries hired" and not just "x_1 = secretaries". (In other examples we will need to include various units such as "x_3 = the number of tons of coal", or "x_4 = the number of board feet of lumber", or "x_5 = the number of hours of machine time".)

Clearly, it does not make sense to hire a negative number of secretaries. That is why we include the nonnegativity condition $x_1 \geq 0$. Likewise, $x_2 \geq 0$. We can combine these as $x_1, x_2 \geq 0$. We will see later that the nonnegativity conditions play a crucial role in the solution process.

Step 2. Obtain the objective function:
In the key sentence that locates the decision variables there is usually the name of some quantity that is to be maximized or minimized. That something is our objective variable. In the current example that quantity is **cost** and we will denote it with C. We must write C in a linear equation involving the decision variables. That is, we must find an equation of the form $C = ax_1 + bx_2$, where the coefficients a and b are determined by the verbal problem in some way. In this example each secretary is paid \$11 per hour. Hence, the cost to the company to hire x_1 secretaries is $\$11x_1$ per hour. Similarly, the cost of the clerks is $\$5x_2$ per hour. Thus, our objective function is $C = 11x_1 + 5x_2$. This gives the total cost per hour to the company. We want to *minimize* the total cost.

Step 3. Determine the constraints:
In each LP problem there are **requirements** which must be met and/or there are **restrictions** which must not be exceeded. These requirement/restrictions are with respect to available time, money, space, labor, or something. Each requirement or restriction will determine a constraint inequality or equality. The requirement (restriction) value will be written on the right hand side of an inequality or equality while the left hand side will be a linear expression involving the decision variables.

In our example, we are required to type (at least) 320 pages per hour. This leads to one constraint. Since x_1 secretaries can type $8x_1$ pages per hour while x_2 clerks can type $4x_2$ pages per hour, our first constraint takes the form $8x_1 + 4x_2 \geq 320$.

A second constraint comes from the requirement that 2600 reports be filed per hour. This leads to the constraint $20x_1 + 40x_2 \geq 2600$.

Step 4. Double check the direction of the constraint inequalities:
It may have seemed in Step 3 that we should have used equalities instead of inequalities. After all, we need 320 pages typed per hour, not more. Why didn't we write $8x_1 + 4x_2 = 320$? because this would have been too restrictive. If we had required that $8x_1 + 4x_2 = 320$ and that $20x_1 + 40x_2 = 2600$, then the only values allowed for x_1 and x_2 would be $x_1 = 10$ and $x_2 = 60$ making $C = 410$. On the other hand, if we hire no secretaries and 80 clerks it will actually be cheaper ($C = 400$) and the inequality constraints will be met. We will have the capability of typing 320 pages and filing 3200 reports. While there may not be 3200 reports to be filed, it is cheaper to pay clerks to have some idle or "slack" time.

Now that we see the merits of having inequality constraints, how do we decide whether they should be $\leq$ or $\geq$? Usually we can determine this without much difficulty if we think about whether we want "at least" ($\geq$) or "at most" ($\leq$). Requirements are usually $\geq$ while restrictions are usually $\leq$.

Here is our completed LP problem for this example:

$$\begin{array}{ll} \text{min} & C = 11x_1 + 5x_2 \\ \text{if} & 8x_1 + 4x_2 \geq 320 \\ & 20x_1 + 40x_2 \geq 2600 \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ \text{where} & x_1 = \text{the number of secretaries to be hired} \\ & x_2 = \text{the number of clerks to be hired} \end{array}$$

EXAMPLE 2:

Fred's Hat Company produces two types of cowboy hats. Each hat of the first type takes Fred 15 minutes to make while each hat of the second type takes him only 10 minutes. Each hat of the first type uses $\frac{1}{2}$ square yard of material while each hat of the second type uses $\frac{3}{4}$ square yard of material. Each work day consists of 7 hours and there are 125 square yards of material on hand each day. If the profit from each type 1 hat is \$9 and the profit on each type 2 hat is \$7, how many hats of each type should Fred make each day to maximize his profit?

Problem formulation: Frequently it is helpful while setting up an LP problem to collect the data in a table. In this case, *the table should be organized by placing the variables from Step 1 as the column headings.* Each constraint will be a row with the restriction or requirement value on the right outside the table.

Step 1. The phrase "how many hats of each type" suggests that we should define the variables as

$x_1 =$ the number of type 1 hats to be made each day.

$x_2 =$ the number of type 2 hats to be made each day.

Note that $x_1 \geq 0$ and $x_2 \geq 0$.

Step 2. Since we want to maximize the profit, we call our objective value P for profit. We list the profit on each type of hat on the bottom of the table we start to make:

	x_1	x_2	
objective	9	7	P

Step 3. Next we enter rows into the table to describe the various constraints. The first constraint reflects the fact that a work day has 7 hours or 420 minutes. Note that we have to use the same units of measurement throughout the problem. That is why we convert hours to minutes. (We could convert minutes to hours instead, but that would mean we have to use even more fractions in the problem.) The values in the columns indicate the amount of time used up in making a hat of the given type. The total amount of time available is placed to the right of the table.

The second constraint reflects the amount of raw materials available each day. The amount used in making one hat of each type is entered into the table and the total amount available is placed to the right of the table.

	x_1	x_2	
time	15	10	420
material	$\frac{1}{2}$	$\frac{3}{4}$	125
objective	9	7	P

Step 4. Since each constraint is a restriction we insert $\leq$ on each constraint:

	x_1	x_2		
time	15	10	$\leq$	420
material	$\frac{1}{2}$	$\frac{3}{4}$	$\leq$	125
objective	9	7		P

Thus our final LP problem is as follows:

$$\begin{array}{ll} \text{max} & P = 9x_1 + 7x_2 \\ \text{if} & 15x_1 + 10x_2 \leq 420 \\ & \frac{1}{2}x_1 + \frac{3}{4}x_2 \leq 125 \\ & x_1, x_2 \geq 0 \\ \text{where} & x_1 = \text{the number of type one hats to be made each day} \\ & x_2 = \text{the number of type two hats to be made each day} \end{array}$$

EXAMPLE 3:

A farm supply dealer makes two type of fertilizer, A_{100} and D_{303} by combining chemicals b and c. Fertilizer A_{100} is made up of 80% chemical b and 20% chemical c. Fertilizer D_{303} is 60% chemical b and 40% chemical c. The dealer needs at least 30 tons of A_{100} and at least 40 tons of D_{303}. She has 100 tons of chemical b and 50 tons of chemical c available. How many tons of each type of fertilizer should she mix in order to maximize the total amount of fertilizer produced?

Problem formulation:

Let x_1 = the number of tons of A_{100} to be made
x_2 = the number of tons of D_{303} to be made

	x_1	x_2		
b available	0.8	0.6	$\leq$	100
c available	0.2	0.4	$\leq$	50
A_{100} needed	1	0	$\geq$	30
D_{303} needed	0	1	$\geq$	40
objective	1	1		T

This problem is more complicated because some of the needed coefficients are not explicitly stated. They turn out to be either 1 or 0 in this particular case. The objective function represents the total amount of fertilizer made and is thus $T = x_1 + x_2$. The requirement of producing at least 30 tons of A_{100} is expressed $x_1 \geq 30$ while the requirement of producing at least 40 tons of D_{303} is expressed $x_2 \geq 40$. Our final model is:

$$\begin{array}{ll} \text{max} & T = x_1 + x_2 \\ \text{if} & 0.8x_1 + 0.6x_2 \leq 100 \\ & 0.2x_1 + 0.4x_2 \leq 50 \\ & x_1 \geq 30 \\ & x_2 \geq 40 \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ \text{where} & x_1 = \text{the number of tons of } A_{100} \text{ to be made} \\ & x_2 = \text{the number of tons of } D_{303} \text{ to be made} \end{array}$$

Note that because of the last two constraints we do not have to explicitly include the nonnegativity conditions. However, it does not hurt to include them and it is probably a good habit to write them down.

Important terms. The following terms were introduced in this chapter. We suggest that you write out the definitions of these and other terms introduced later and keep them all in one place for easy reference.

linear programming problem
objective function
constraint
nonnegativity conditions